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Tauberian Conditions for a Class of Matrices

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1

Let $A = (a_{mn})$ be a positive regular matrix; by $\mathbf{A}$ we denote the set of sequences limited by A and by $\mathbf{A}_0$ those sequences limited to zero by A. If $\{\xi_n\}$ is a bounded sequence and $\{\xi_n S_n\} \in \mathbf{A}_0$ whenever $\{S_n\} \in \mathbf{A}_0$, then $\{\xi_n\} = \xi$ is called an f-sequence for the matrix $A = (a_{mn})$. The set of f-sequences will be denoted by $\mathbf{A}^0$. The set $\mathbf{A}^0$ should not be confused with $\mathbf{A}^*$, the set of factor sequences for $A = (a_{mn})$. A factor sequence, [5, 7], is defined in terms of bounded sequences $\{S_n\}$, whereas here we place no restriction on $\{S_n\}$. In this paper we shall use f-sequences as a means of establishing Tauberian conditions for regular matrices. The method developed applies to any regular matrix but in this paper it is applied only to Riesz and Nørlund means. The conditions obtained for these means in Section 5 are already well known, see [8], but the present approach is easy and efficient.

If ξ and ζ both belong to $\mathbf{A}^0$, then it is clear that $\{\xi_n(\zeta_n S_n)\} \in \mathbf{A}_0$ for all $\{S_n\} \in \mathbf{A}_0$. From this it follows that $\xi\zeta$ is in $\mathbf{A}^0$ and it is also easy to show that $a\xi + b\zeta \in \mathbf{A}^0$ where a and b are scalars. The unit sequence $u = 1, 2, \ldots$ also belongs to $\mathbf{A}^0$.

If $\mathbf{B} \supset \mathbf{A}$, then the matrix $B = (b_{m,n})$ is said to be a-stronger than the matrix $A = (a_{m,n})$, if $\mathbf{B}_0 \supset \mathbf{A}_0$ we shall say $B = (b_{m,n})$ is c-stronger than $A = (a_{m,n})$. If $\mathbf{B} \supset \mathbf{A}$ then every bounded sequence limited to zero by $A = (a_{m,n})$ is limited to zero by $B = (b_{m,n})$, see [6, p. 85]. However, $\mathbf{B} \supset \mathbf{A}$ does not imply $\mathbf{B}_0 \supset \mathbf{A}_0$, see [6, p. 86].

We prove:

THEOREM 1. *Let $A = (a_{m,n})$ be a positive regular matrix and ξ a bounded* *sequence such that* $c_2 \geqslant \xi_n \geqslant c_1 > 0$; *then* ξ *is an f-sequence if and only if* $\mathbf{B}_0 \supset \mathbf{A}_0$, *where*

$$b_{m,n} = a_{m,n}\xi_n \Big/ \left(\sum_{n=1}^{\infty} a_{m,n}\xi_n \right) \tag{1}$$

$(m, n = 1, 2, ...)$.

Proof. We note

$$c_1 \sum_{n=1}^{\infty} a_{m,n} \leqslant \left| \sum_{n=1}^{\infty} a_{m,n}\xi_n \right| \leqslant c_2 \sum_{n=1}^{\infty} a_{m,n} \tag{2}$$

so that if $s \in \mathbf{A}_0$ and ξ is an f-sequence, then $s \in \mathbf{B}_0$. On the other hand, $\mathbf{B}_0 \supset \mathbf{A}_0$ implies $\xi s \in \mathbf{A}_0$ whenever $s \in \mathbf{A}_0$.

If ξ is an f-sequence, then for some real number a, there exist $c_1 > 0$, $c_2 > 0$ such that $c_1 \leqslant au_n + \xi_n \leqslant c_2$, $(n = 1, 2, ...)$. Hence, when testing a vector space including the unit sequence for f-sequences, we need only look at those satisfying such a condition.

2

A triangular matrix $A = (a_{m,n})$ is called an M'-matrix if for some $\kappa > 0$,

$$\left| \sum_{k=1}^{n} a_{m,k} S_k \right| \leqslant \kappa \left| \sum_{k=1}^{n'} a_{n',k} S_k \right|$$

for some n', $n' = n'(n)$ $(0 \leqslant n' \leqslant n)$ and for all m. The number n' depends on n and $\{S_n\}$ but is independent of m.

The concept of an M matrix extends to non-regular matrices. It may turn out that not only is $A = (a_{m,n})$ an M' matrix but that $C = (c_{m,n})$ is an M matrix, where

$$c_{m,n} = f(m)\, a_{m,n} \qquad \text{and} \qquad f(m) \uparrow \infty.$$

Such a function $f(m)$ we shall call a *regulating* function. The following theorem may be found in [4, 6]; see also [1, 8].

THEOREM 2. *Let* $f(m)$ *be a regulating function for the positive regular and triangular* M *matrix* $A = (a_{m,n})$. *If* $B = (b_{m,n})$ *is a second regular triangular matrix such that,*

$$\frac{1}{f(m)} \sum_{n=1}^{m} f(n) \left| \frac{b_{m,n}}{a_{m,n}} - \frac{b_{m,n+1}}{a_{m,n+1}} \right| \leqslant M \tag{3}$$

$(b_{m,m+1}/a_{m,m+1} = 0$ *by convention*) *then* $\mathbf{B}_0 \supset \mathbf{A}_0$.

We now prove:

THEOREM 3. *Let $f(m)$ be a regulating function for the positive regular triangular M matrix $A = (a_{m,n})$. Then if*

$$\xi_{n+1} - \xi_n = O((f(n+1) - f(n))/f(n)),$$

then ξ is an f-sequence for A.

Proof. Let $B = (b_{m,n})$ be defined as in (1), then we must show (3) to be true. Since (2) may be assumed, we must in effect show

$$\frac{1}{f(m)} \sum_{n=1}^{m} f(n) \left| \frac{a_{m,n}\xi_n}{a_{m,n}} - \frac{a_{m,n+1}\xi_{n+1}}{a_{m,n+1}} \right| \leqslant M$$

but this is true if

$$\frac{1}{f(m)} \sum_{n=1}^{m} f(n) \mid \xi_{n+1} - \xi_n \mid \leqslant M.$$

This last statement follows immediately from the hypothesis.

3

Suppose $\{t_n\}$ is a bounded sequence such that $t_n = 0$, $n \neq n_k$ where

$$\lim_{m\to\infty} \sum \mid a_{m,n_k} \mid = 0,$$

then t is called a *thin* sequence. Let ξ be an f-sequence whenever

$$\mid \xi_n - \xi_{n+1} \mid = O(g(n)),$$

where $\lim_{n\to\infty} g(n) = 0$. Further, let $S \in \mathbf{A}_0$ and $\mid S_n - S_{n+1} \mid = O(g(n))$.

If S is bounded and A is a positive matrix, then $S = r + t$ where r is convergent to zero and t is thin, see [5]. If S is unbounded, we define

$$\xi_n = 1/S_n \qquad \text{whenever} \quad \mid S_n \mid \geqslant 1$$

and

$$\xi_n = S_n \qquad \text{whenever} \quad \mid S_n \mid < 1.$$

If $S_k \geqslant 1$, and $\frac{1}{2} < S_{k+1} < 1$, then

$$\mid \xi_k - 1 \mid \leqslant \left| \frac{1}{S_k} - 1 \right| \leqslant \left| \frac{S_k - 1}{S_k} \right| \leqslant \left| \frac{S_k - S_{k+1}}{S_k} \right| \leqslant Mg(k);$$

$$\mid 1 - \xi_{k+1} \mid \leqslant \mid 1 - S_{k+1} \mid \leqslant \mid S_k - S_{k+1} \mid \leqslant Mg(k);$$

and $|\xi_k - \xi_{k+1}| \leqslant 2Mg(k)$. A similar investigation for the other possible combinations of values of S_k, S_{k+1} shows that $|\xi_k - \xi_{k+1}| = O(g(k))$. It follows that $\xi S \in \mathbf{A}_0$, but $\xi_n S_n = 1$ whenever $|S_n| \geqslant 1$, $\xi_n S_n = S_n{}^2$ whenever $|S_n| < 1$. It follows that if A is a positive matrix ξS can be expressed in the form $r + t$ where r converges to zero and t is thin. It then follows after some consideration that S can be expressed in this form.

Before leaving this proof, we observe that the set of factor sequences form a Banach Algebra and this fact is used in proving that if S is both a factor sequence and is limited to zero, then $s = r + t$. The set of f-sequences is not necessarily complete. Suppose S converges to zero; let $S_n{}^k = S_n$, $n \leqslant k$, $S_n{}^k = S_k$, $n > k$. Then

$$\sup_n |S_n{}^i - S_n{}^j| = \| S^i - S^j \| < \epsilon,$$

$i, j > N$. Also, $\| S^i - S \| < \epsilon$ for $i > N$. However, if σ is an unbounded sequence limited to zero by A, $S^k\sigma$ will always be limited to zero ($k = 1, 2, \ldots$) but if $|\sigma_n|$ is not limited and $S = \{\text{sign}\ \sigma_n/(\sigma_n)^{1/2}\}$ then $S\sigma$ is not limited to zero by A.

4

We have seen that if S is an f-sequence for a regular matrix $A = (a_{m,n})$ and $S \in \mathbf{A}$, then $S = r + t$ where r is convergent and t is thin. For converse results see [2, 3]. Thus, if the set of f-sequences is known we often have a good starting point for finding Tauberian conditions for the matrix. Before applying this technique to some examples, however, it is well to look at one case where the technique does not work.

Let $A = (a_{m,n})$ be the matrix defined by the transformations

$$t_n = \tfrac{1}{2}(S_{2n} + S_{2n+1}),$$

($n = 1, 2, \ldots$). Then A limits any sequence satisfying $S_{2n} = -S_{2n+1}$ to zero. Suppose ξ is an f-sequence and

$$\xi_{2n} - \xi_{2n+1} = \epsilon_{2n},$$

then

$$\xi_{2n}S_{2n} + \xi_{2n+1}S_{2n+1} = \xi_{2n}(S_{2n} + S_{2n+1}) + (\xi_{2n+1} - \xi_{2n})\, S_{2n+1}\,.$$

Since for all bounded ξ,

$$\lim_{n\to\infty} \xi_{2n}(S_{2n} + S_{2n+1}) = 0,$$

we need only choose S so that

$$\lim_{n\to\infty} \epsilon_{2n} S_{2n+1}$$

does not exist in order to show ξ is not an f-sequence. This is always possible unless $\epsilon_{2n} = 0$, $n > N$. From this we would conclude that $S_{2n} = S_{2n+1}$, $n > N$ is a Tauberian condition for A since there are clearly no thin sequences. On the other hand,

$$\lim_{n\to\infty} | S_{2n} - S_{2n+1} | = 0,$$

is a Tauberian condition for A, and the technique is in this case very inefficient.

5

Let $R(p_n)$ be the Riesz mean defined by

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k S_k ,$$

where $P_n = \sum_{k=1}^{n} p_k$, $p_k > 0$ $(k = 0, 1, 2,...)$, $P_n \uparrow \infty$. Keeping the same notation and restrictions on p_n and P_n, the Nørlund mean is given by

$$\mathbf{T}_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k .$$

The Riesz mean is an M matrix and the Nørlund mean is an M matrix if the additional conditions $p_n \leqslant p_{n-1}$ and

$$p_n / p_{n-1} \geqslant p_{n-1} / p_{n-2}$$

$(n = 1, 2,...)$ are satisfied, see [6, p. 76]. For both types of matrices $f(n) = P_n$ is a regulating function and ξ is an f-sequence if

$$| \xi_n - \xi_{n+1} | = O\left(\frac{p_n}{P_n}\right),$$

as in Theorem 3.

Let $R(p_n)$ be defined by $P_n = e(\mu_n)$, $[e^x = e(x)]$ where

$$\mu_n = \sum_{k=1}^{n} \frac{1}{\lambda_k}, \qquad \sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty, \qquad \lambda_n \downarrow 0.$$

Then, we have

$$\frac{p_n}{P_n} = 1 - e(\mu_{n-1} - \mu_n) = 1 - e\left(\frac{-1}{\lambda_n}\right)$$
$$= \frac{1}{\lambda_n} - \frac{1}{2!}\left(\frac{1}{\lambda_n}\right)^2 + \cdots + \frac{(-1)^{k+1}}{k!}\left(\frac{1}{\lambda_n}\right)^k + \cdots \leqslant \frac{1}{\lambda_n}.$$

On the other hand, we have

$$\frac{1}{P_n}\sum_{r=n-k}^{n} p_r = \frac{P_n - P_{n-k-1}}{P_n} = 1 - e(\mu_{n-k-1} - \mu_m)$$
$$\geqslant \frac{1}{2}\left(\frac{1}{\lambda_{n-k}} + \cdots + \frac{1}{\lambda_n}\right).$$

Hence,

$$\frac{1}{P_n}\sum_{r=n-k}^{n} p_r \geqslant \frac{1}{2}\sum_{r=n-k}^{n}\frac{p_r}{P_r}$$

and from this it follows that if

$$| S_n - S_{n-1} | = O\left(\frac{p_n}{P_n}\right) \qquad \text{then} \quad | S_n - S_{n-k} | > 1,$$

implies $1/P_n \sum_{r=n-k}^{n} p_r > \frac{1}{4}$ and S is not thin. Hence $O(p_n/P_n)$ is a Tauberian condition for $R(p_n)$.

Let $N(p_n)$ be the Nørlund mean defined by $p_n = n^{-1/2}$. Then $p_n/P_n = O(1/n)$ and if

$$| S_{n-k} - S_n | > \epsilon_0 > 0,$$

then $k \geqslant cn$ for some $c > 0$. Also, we have

$$\sum_{r=n-k}^{n} | a_{n,r} | = \frac{1}{P_n}\sum_{r=0}^{k} p_r .$$

But

$$P_n \leqslant 1 + \int_1^n \frac{dx}{(x)^{1/2}} \leqslant 2(n)^{1/2}$$

and

$$\sum_{r=0}^{k} p_r \geqslant \int_1^{k+1} \frac{dx}{(x)^{1/2}} \geqslant 2(k+1)^{1/2} - 2,$$

so that if $k \geqslant cn$

$$\frac{1}{P_n}\sum_{r=0}^{k} p_r \geqslant \frac{2c_1(n)^{1/2} - 2}{2(n)^{1/2}} \geqslant \frac{c_1}{2}, \qquad n > N.$$

From this we conclude that if S is an f-sequence it is convergent and $O(1/n)$ is a Tauberian condition for $N(p_n)$. A similar analysis shows $O(1/n)$ to be a Tauberian condition for $N(p_n)$ generated by $p_n = n^\alpha$, $-1 < \alpha \leqslant 0$.

Using the techniques developed in (2), it could be shown in most cases that the conditions obtained here for Riesz and Nørlund means are best possible. We have already seen that the class of f-sequences always determines a Tauberian conditions, though the example in Section 4 shows that for some matrices the Tauberian conditions derived may be trivial and uninteresting. If it were possible to find sufficient conditions on the matrix for the method of f-sequences to give the best possible conditions, then the value of the technique would be greatly increased.

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